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Selected topics of nonlinear analysis-exercises

Pozna 2015

CHAPTER **1**

Fixed point theorems

Exercise 1 ([10, p. 17, 1.6.3])**.** Let $f\colon X \to X$ be a mapping of an arbitrary nonempty set X into itself. Show that if $f^n := \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$ has exactly one fixed point for some $n \in \mathbb{N}$, then so does f.

Exercise 2 ([10, p. 17, 1.6.1])**.** (a) Prove that the assumption of completeness in the Banach Contraction Principle cannot be omitted.

(b) Prove that the condition $d(f(x), f(y)) \leq \alpha d(x, y), \alpha < 1$" in the B.C.P. cannot be relaxed to $d(f(x), f(y)) < d(x, y)$ for $x \neq y$".

Exercise 3. Show that (b) in the previous exercise is true even in case of a bounded space.

Exercise 4 ([10, p. 17, 1.6.1])**.** Show that if $\langle X, d\rangle$ is a compact metric space and $f\colon X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ for $x \neq y$, then f has a unique fixed point.

Exercise 5 ([10, p. 18, 1.6.8])**.** Let $\langle X, d\rangle$ be complete and $f\colon X \to X$ surjective and expanding (that is, there exists a $\beta > 1$ such that $d(f(x), f(y)) \geq \beta d(x, y)$ for $x, y \in X$). Show that f is bijective and has a unique fixed point z with $f^{-n}(u) := (f^{-1})^n(u) \to z$ for each $u \in X$.

Exercise 6. Show that there exists a compact metric space X, a pair of points $a, b \in X$, a constant $\alpha \in (0, 1)$ and a fixed-point free mapping $f\colon X \to X$ such that: $d(f(x), f(y)) \leq \alpha d(x, y)$ unless $\{x, y\} \neq \{a, b\}$ and $d(f(a), f(b)) = d(a, b)$.

Exercise 7 ([10, p. 25, 2.2.1])**.** Let $K\colon [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy a contraction condition in the third variable: $|K(t, s, x) - K(t, s, y)| \leq \alpha|x - y|$ for all $s, t \in [0, 1]$, $x, y \in \mathbb{R}$, where $\alpha \in (0, 1)$. Prove that for any $v \in C[0, 1]$, the nonlinear Volterra integral equation

$$u(t) = v(t) + \int_0^t K\big(t, s, u(s)\big)ds, \qquad t \in [0, 1], \tag{7.1}$$

has a unique solution $u \in C[0, 1]$. Moreover, for any $u_0 \in C[0, 1]$, the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n(t) := v(t) + \int_0^t K\big(t, s, u_{n-1}(s)\big)ds$ for $n \in \mathbb{N}$, converges to this solution, uniformly on $[0, 1]$.

Exercise 8 ([10, p. 2, 1.3.1]). Show that any contraction $f\colon X \to X$, where (X,d) is a metric space, satisfies the condition

$$\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0\ \forall x \in X\colon\ d(x, f(x)) < \delta \implies f(B(x,\varepsilon)) \subset B(x,\varepsilon). \tag{8.1}$$

Exercise 9 ([10, p. 12, 1.3.1]). (a) Show that there exists a complete metric space X and a mapping $f\colon X \to X$ satisfying (8.1), having a fixed point, but not continuous.

(b) Prove that any mapping satisfying (8.1) is continuous at any fixed point.

Exercise 10 (a classical theorem). Prove that every continuous mapping $F\colon [0,1] \to [0,1]$ has a fixed point.

Exercise 11. Prove that the system of equations

$$\begin{cases} |xy| - x = 0 \\ 2y^2 - 1 = \sin(x+y) \end{cases} \tag{11.1}$$

has a solution.

Exercise 12 ([15, p. 590]). Use the Schauder fixed point theorem to prove that the closed unit ball $\overline{B}$ in $C[-1,1]$ is not compact.

Exercise 13. Let $X = C := [-1,1]$, $A := \{-1,1\}$, $f\colon X \to C\colon x \mapsto -x$. Show that $f \in \mathcal{K}_A(X,C)$ and that f is essential.

Exercise 14 ([10, p. 72, 4.9.34]). Let $p\colon E \to \mathbb{R}_+$ be a function defined on a normed space E such that $p^{-1}(\{0\}) = \{0\}$ and $p(\lambda x) = \lambda p(x)$ for $\lambda > 0$. Let $C \subset E$ be convex, $U \subset C$ open and such that $0 \in U$ and $f\colon \bar{U} \to C$ compact. Assume that for $x \in \partial U$ any of the following conditions is satisfied:

(a) $p(f(x)) \le p(x)$;

(b) $p(f(x)) \le p(f(x) - x)$;

(c) $p(f(x)) \le \sqrt[k]{\big(p(x)\big)^k + \big(p(f(x)-x)\big)^k}$ for some $k > 1$.

Show that f has a fixed point.

Exercise 15 ([10, p. 49, 3.8.16]). Let X be a fixed-point space. Prove that every retract of X is also a fixed point space.

Exercise 16 ([10, p. 49, 3.8.19]). Show that the closed unit ball $\overline{B}$ in l^2 is not a fixed-point space.

Chapter 2

Measures of noncompactness

Exercise 17. (a) [10, p. 55, Proof of 4.2.3] Show that if a subset A of a metric space X is relatively compact, then for any $\varepsilon > 0$ there exists a finite set $\{c_1, \ldots, c_n\} \subset A$ such that $A \subset \bigcup_{k=1}^{n} B(c_k, \varepsilon)$.

(b) Prove, that any nonempty subset of a complete metric space is relatively compact if and only if it is totally bounded.

Exercise 18. Let A be a subset of a norm space and let $\delta(\cdot)$ denote the diameter of the corresponding set. Prove that:

(a) $\delta(\bar{A}) = \delta(A)$;

(b) $\delta(\lambda A) = |\lambda|\delta(A)$, where λ is an element of a given field;

(c) $\delta(\operatorname{conv} A) = \delta(A)$, where $\operatorname{conv} A$ denotes the convex hull of the set A.

Exercise 19. Let γ denote the measure of noncompactness α or β. Prove that for any subsets A, B of a Banach space E, the properties 7^0, 8^0 and 9^0 hold.

Exercise 20. Prove that a metric d on a linear space V over the field $\mathbb{R}$ is determined by a norm $\|\cdot\|$ if and only if that metric is invariant in view if a translation and absolutely homogeneous.

Exercise 21. Let V be a vector space over $\mathbb{R}$ with a translation invariant metric d. Check that: $\delta(A + B) \leq \delta(A) + \delta(B)$ for any sets $A, B \subseteq V$.

Exercise 22. Let us consider $\mathbb{R}^2$ with the radial metric which for $z_1, z_2 \in \mathbb{R}^2$ is defined by the following formula

$$d_p(z_1, z_2) = \begin{cases} \rho(z_1, z_2) & \text{if } \theta, z_1, z_2 \text{ are colinear,} \\ \rho(z_1, \theta) + \rho(z_2, \theta) & \text{otherwise,} \end{cases}$$

where ρ denotes the Euclidean metric and $\theta = (0, 0)$. Check that

(a) the metric d_p is absolutely homogenous;

(b) the metric space $(\mathbb{R}^2, d_p)$ is complete;

(c) the metric d_p does not come from any norm;

(d) the topology generated by the metric d_p is not linear.

Exercise 23. Let us consider the metric ,,river" which for $v_1 = (x_1, y_1), v_2 = (x_2, y_2) \in \mathbb{R}^2$ is defined by the following formula

$$d_r(v_1, v_2) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |y_1| + |y_2| + |x_1 - x_2|, & \text{if } x_1 \neq x_2. \end{cases}$$

Check that

(a) the metric d_r is absolutely homogeneous;

(b) the metric d_r does not come from any norm;

(c) the metric space $(\mathbb{R}^2, d_r)$ is complete;

(d) the topology generated by the metric d_r is not linear.

Exercise 24. Check that the topology on $\mathbb{R}^2$ determined by the radial metric and the topology determined by the metric ,,river" are not comparable.

Exercise 25 ([8, p. 399, Example 1])**.** Let the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the formula $f(x, y) := \left(\frac{x+1}{2}, y\right)$. Prove that the function f is continuous in the metric ,,river", but it is not continuous in the radial metric.

Exercise 26. Let the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the formula

$$f(x, y) := \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Prove that the function f is continuous in the radial metric, but it is not continuous in the metric ,,river".

Exercise 27. Let X be an infinite set. Prove that the Kuratowski measure of noncompactness of any nonempty subset of a metric space (X, d), where d is a discrete metric, can be expressed by the following formula

$$\alpha(A) = \begin{cases} 1, & \text{if } A \text{ is an infinite set}, \\ 0, & \text{if } A \text{ is a finite set}. \end{cases}$$

Exercise 28. Check that

(a) the properties 4° and 8° of the measure α do not have to be satisfied in vector spaces endowed with a metric which is not absolutely homogeneous;

(b) in a vector space endowed with a metric which is not absolutely homogeneous, the following inequality holds

$$h\alpha(A) \leq \alpha\Big(\bigcup_{0\leq\lambda\leq h} \lambda A\Big), \quad h \geq 0,$$

where A is any bounded subset of that space; check that even in such spaces the inverse inequality does not have to hold;

(c) the property 6° is satisfied in a vector space endowed with a metric which is not translation invariant, while it does not have to be satisfied in vector spaces endowed with a metric which is not translation invariant;

(d) the property 7° does not have to be satisfied neither in vector spaces endowed with a metric which is absolutely homogeneous nor in vector spaces endowed with a metric which is translation invariant.

Exercise 29 ([7, p. 178, Theorem 2]). Let us consider the space $\mathbb{R}^2$ with the metric „river". Let D be its bounded subset. Let us introduce the following notation.

- We say that a number $y \in \mathbb{R}$ satisfies the condition $A^*(D)$, if for every $\varepsilon > 0$ there exists infinitely many points in D which have pairwise different abscissas and ordinates of which belong to the interval $(y - \varepsilon, y]$.
- We say that a number $y \in \mathbb{R}$ satisfies the condition $A_*(D)$, if for every $\varepsilon > 0$ there exists infinitely many points in D which have pairwise different abscissas and ordinates of which belong to the interval $[y, y + \varepsilon)$.
- Let $y^*(D)$ denotes the supremum of absolute values of numbers satisfying at least one of the above conditions (or zero, if does not exist any such number).

(a) Prove that if does not exist a number satisfying at least one of the conditions $A^*(D)$, $A_*(D)$, then the set D consists of finitely many parts such that each of those parts is included in a vertical line, so it is compact.

(b) Prove that in the opposite case $\alpha(D) \geq 2y^*(D)$.

(c) Let $R_{a,b}$, where $a \in \mathbb{R}$, $b > 0$, be a rectangle $[a - b, a + b] \times [-b, b]$. Prove that $\alpha(R_{a,b}) = 2b$.

(d) Prove that $\beta(D) \leq y^*(D)$.

(e) Deduce from the above items that $\alpha(D) = 2y^*(D)$ and $\beta(D) = y * (D)$.

Exercise 30 ([7, p. 179, Theorem 4]). Let us consider the space $\mathbb{R}^2$ with the radial metric. Let D be its bounded subset. Let us introduce the following notation.

- We say that a number $w \in \mathbb{R}$ satisfies the condition $W^*(D)$, if for every $\varepsilon > 0$ there exists infinitely many points v in D, satisfying the condition $w - \varepsilon < \|v\|_2 \leq w$ and belonging to pairwise different lines passing through the origin.

- We say that a number $w \in \mathbb{R}$ satisfies the condition $W_*(D)$, if for every $\varepsilon > 0$ there exist infinitely many points v in D satisfying the condition $w \leq \|v\|_2 < w + \varepsilon$ and belonging to pairwise different lines passing through the origin.
- Let $w^*(D)$ denotes the supremum of absolute values of numbers satisfying at least one of the above conditions (or zero, if does not exist).

(a) Prove that if does not exist a number satisfying at least one of the conditions $W^*(D)$, $W_*(D)$, then the set D consists of finitely many parts such that each of those parts is included in a line passing through the origin, so it is compact.

(b) Prove that in the opposite case $\alpha(D) \geq 2w^*(D)$.

(c) Prove that $\beta(D) \leq w^*(D)$.

(d) Deduce from the above items that $\alpha(D) = 2w^*(D)$ oraz $\beta(D) = w * (D)$.

Exercise 31. Prove the Ambrosetti Lemma for functions admitting values in a vector space endowed with a translation invariant metric, that is, to prove that if J is a compact subset of a metric space (X, ρ), V is a vector space endowed with a translation invariant metric d, and H an equicontinuous and uniformly bounded family of functions $h\colon J \to V$, then $\alpha\left(H(J)\right) = \max_{t \in J} \alpha(H(t))$, where $H(J) = \{h(J) : h \in H\}$ and $H(t) = \{h(t) : h \in H\}$ for $t \in J$.

Exercise 32 ([13, p. 174, Ćwiczenie 8])**.** Prove the following generalization of the Cantor theorem. Let F_t, $t \in T$, be a family of closed sets in a complete metric space, satisfying the following conditions:

(i) the intersection of a finite number of sets F_t is nonempty;

(ii) $\inf_{t \in T} \alpha(F_t) = 0$.

Then $\bigcap_{t \in T} F_t \neq \emptyset$.

Exercise 33. Give an example of a mapping which satisfies the assumptions of the Sadovski Fixed Point Theorem but does not satisfy the assumptions of the Darbo Fixed Point Theorem.

Chapter 3

Hyperconvex metric spaces

Exercise 34 ([11, s. 394, Remark after Definition 2.3])**.** Let (X, d) be a metric space. Prove that the following conditions are equivalent:

(a) the space X is totally convex;

(b) if $d(x, y) = r + s$, then the closed balls $\overline{B}(x, r)$ i $\overline{B}(y, s)$ intersect;

(c) if $d(x, y) \leq r + s$, then the closed balls $\overline{B}(x, r)$ i $\overline{B}(y, s)$ intersect.

Exercise 35 ([2, p. 407, Remark 1])**.** Prove that if $T\colon X \to Y$ is a uniformly continous mapping from a totally convex space X to the metric space Y, then its minimal modulus of continuity is subadditive.

Exercise 36 ([5, s. 20, Lemma 3.1.14])**.** Let X, Y be metric, $A \subset X$ and $B \subset Y$ be nonempty sets and $T\colon X \to B$ be a mapping of modulus of continuity ω. Prove that there exists a maximal extension $\tilde{T}$ of the mapping T having the same modulus of continuity and and mapping and mapping every point outside the domain of the mapping T into the set B.

Exercise 37 ([5, s. 31, Proposition 4.2.5])**.** Prove that if A is a subset of a totally convex space X, $x \in X \setminus A$ and $a \in A$ is such that $d(x, a) = d(x, A)$, then a belongs to the boundary of the set A.

Exercise 38 ([5, s. 30, Example 4.1.1])**.** Prove that every closed interval included in the real line is hyperconvex.

Exercise 39. Prove that the space $\mathbb{R}^n$ endowed with the Euclidean metric is hyperconvex if and only if $n = 1$.

Exercise 40 ([5, s. 17])**.** Prove that if a metric space is hyperconvex, then it possesses the property (P), but not necessary conversely.

Exercise 41 ([16, s. 66, Remark 1.2])**.** Prove that the hyperconvexity of the space X is equivalent to the following property: for any function $r\colon X \to [0, +\infty)$ satisfying the inequality $d(x, y) \leq r(x) + r(y)$ for any $x, y \in X$, there exists such a point $z \in X$ that $d(x, z) \leq r(x)$ for every $x \in X$.

Exercise 42 ([5, s. 30, Proposition 4.1.2])**.** Prove that every hyperconvex space is complete.

Exercise 43. Prove that every bounded subset A of a hyperconvex space is included in a certain ball of radius $\frac{1}{2}\operatorname{diam} A$.

Exercise 44 ([5, s. 34, Example 4.3.1])**.** Check that the intersection of two hyperconvex spaces does not have to be hyperconvex.

Exercise 45 ([5, s. 26, Theorem 3.2.5])**.** Prove that a nonexpansive retract of a hyperconvex space is hyperconvex.

Exercise 46 ([5, s. 26, Example 3.2.6])**.** Prove that the image of a hyperconvex space under a nonexpansive mapping does does not have to be hyperconvex.

Exercise 47 ([5, s. 39, Remark 4.3.9])**.** Give a chain of hyperconvex spaces which has an empty intersection.

Exercise 48 ([5, s. 41, Theorem 4.4.1])**.** Prove that the product of two hyperconvex spaces endowed with the maximum metric is hyperconvex.

Exercise 49 ([5, s. 50, Theorem 4.5.8])**.** Give an example of a linear subspace of the space $\mathbb{R}^3$ endowed with the ,,maximum" norm, which is not hyperconvex.

Exercise 50 ([5, s. 55, Example 4.6.4])**.** Prove that a metric segment is a hyperconvex hull of two point space.

Exercise 51 ([9, s. 334, 1.16])**.** Construct a hyperconvex hull of three point space.

Exercise 52 ([5, p. 63, Example 4.6.21])**.** Prove that the hyperconvex hull of c_0 in l^∞ is the whole space l^∞.

Exercise 53 ([5, s. 76, Corollary 5.2.3])**.** Prove that any continuous mapping of a hyperconvex space into itself possesses a fixed point.

Exercise 54 ([11, s. 397, Remark after Definition 3.4])**.** Prove that a nonempty intersection of admissible sets is admissible.

Exercise 55 ([11, s. 398, Theorem 3.10])**.** Prove that an admissible subset of a hyperconvex space H is hyperconvex.

Exercise 56 ([5, s. 41, Proposition 4.2.6])**.** Prove that if A is an admissible subset of a hyperconvex space H, then every point of the space H possesses the nearest point in A.

Exercise 57 ([5, s. 43, Proposition 4.2.10])**.** Let H be a hyperconvex space, $A = \bigcap_{\lambda\in\Lambda} \overline{B}(x_\lambda, r_\lambda)$ be an admissible set and let $r > 0$. Prove that $\bigcup_{x\in A} \overline{B}(x, r) = \bigcap_{\lambda\in\Lambda} \overline{B}(x_\lambda, r_\lambda + r)$.

Exercise 58 ([5, s. 36, Lemma 4.3.7])**.** Prove that the intersection of a chain of admissible subsets of a hyperconvex space is nonempty.

Chapter **4**

Functions of bounded variation

Exercise 59 ([1, Proposition 1.3(d), p. 56])**.** Prove that if $f\colon [a,b]\to\mathbb{R}$ is a function of bounded variation in the sense Jordan, then it is bounded and that $\|f\|_\infty \le |f(a)| + \operatorname{var}(f;[a,b])$.

Exercise 60 ([1, Proposition 1.3(a), p. 56])**.** Prove that if $f,g\colon [a,b]\to\mathbb{R}$ are functions of bounded variation in the sense of Jordan, then also $f+g$ is a function of bounded variation in the sense of Jordan and that the following inequality holds $\operatorname{var}(f+g;[a,b]) \le \operatorname{var}(f;[a,b]) + \operatorname{var}(g;[a,b])$.

Exercise 61 ([1, Proposition 1.3(e), p. 56])**.** Prove that every monotone function $f\colon [a,b]\to\mathbb{R}$ is of bounded variation in the sense of Jordan and $\operatorname{var}(f;[a,b]) = |f(b)-f(a)|$.

Exercise 62 ([1, Proposition 1.10, p. 62])**.** Prove that if $f,g\colon [a,b]\to\mathbb{R}$ are functions of bounded variation in the sense of Jordan, then

$$\operatorname{var}(fg;[a,b]) \le \|f\|_\infty \operatorname{var}(g;[a,b]) + \|g\|_\infty \operatorname{var}(f;[a,b]).$$

Exercise 63 ([1, Proposition 1.3(g), p. 56])**.** Prove that the variation in the sense of Jordan is an additive function of an interval, that is the following equality holds

$$\operatorname{var}(f;[a,b]) = \operatorname{var}(f;[a,c]) + \operatorname{var}(f;[c,b]),\ c\in[a,b],$$

where $f\colon [a,b]\mapsto\mathbb{R}$ is a function of bounded variation in the sense of Jordan.

Exercise 64 ([1, Exercise 1.1, p. 104])**.** Prove that if functions $f,g\colon [a,b]\to\mathbb{R}$ are functions of bounded variation in the sense of Jordan and $m := \inf_{x\in[a,b]}|g(x)| > 0$, then the quotient f/g is also a function of bounded variation in the sense of Jordan.

Exercise 65 ([12, p. 60])**.** Prove that if a function $f\colon [a,b]\to\mathbb{R}$ satisfies the Lipschitz condition [1], then it is of bounded variation in the sense of Jordan and that $\operatorname{var}(f;[a,b]) \le L(b-a)$. Does any function of bounded variation in the sense of Jordan have to satisfy a Lipschitz condition?

[1] Let us recall that a function $f\colon [a,b]\to\mathbb{R}$ satisfies the Lipschitz condition, if there exists such a constant $L\ge 0$ that $|f(x)-f(y)| \le L|t-s|$ for any $x,y\in[a,b]$. Let us notice that functions satisfying the Lipschitz condition are uniformly continuous.

Exercise 66 ([1, Exercise 1.3, p. 104])**.** Prove that if $f\colon [a,b] \to \mathbb{R}$ is a function of bounded variation in the sense of Jordan, then also $|f|$ is a function of bounded variation in the sense of Jordan and that the following inequality holds

$$\operatorname{var}(|f|;[a,b]) \leq \operatorname{var}(f;[a,b]).$$

Exercise 67 ([1, Exercise 1.4, p. 104])**.** Show an example of such a function $f\colon [0,1] \to \mathbb{R}$ of unbounded variation in the sense of Jordan that $|f|$ is a function ofbounded variation in the sense of Jordan.

Exercise 68 ([1, Example 1.4, p. 58])**.** Show an example of a function of bounded variation in the sense of Jordan, which is not monotone on every interval.

Exercise 69 ([14, Exercise 2.3, p. 41])**.** Prove that if $f\colon [a,b] \to \mathbb{R}$ is a function of C^1–class, then

$$\operatorname{var}(f;[a,b]) = \int_a^b |f'(t)|\mathrm{d}t. \tag{69.1}$$

Exercise 70 ([12, p. 61])**.** Let $g\colon [a,b] \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. Evaluate the variation in the sense of Jordan of the function $f\colon [a,b] \to \mathbb{R}$ given by the formula

$$f(x) = c + \int_a^x g(t)\mathrm{d}t.$$

Exercise 71. Let us consider the function $f\colon [0,1] \to \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ x^3 \sin \frac{1}{x}, & \text{if } x \in (0,1]. \end{cases}$$

Show that $\frac{2}{3} \leq \operatorname{var}(f;[0,1]) \leq \frac{3}{2}$.

Exercise 72 ([1, Example 1.8])**.** Show that the function $f\colon [0,1] \to \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{for } x \in (0,1], \\ 0 & \text{for } x = 0, \end{cases}$$

is continuous, but $f \notin BV[0,1]$.

Exercise 73 ([1, Exercise 1.15])**.** For a given function $f\colon [a,b] \to \mathbb{R}$ let $v_f(x) = \operatorname{var}(f;[a,x])$ for $x \in [a,b]$. Find the formula for the function v_f, if $f(x) = \sin x$ and $[a,b] = [0,2\pi]$.

Exercise 74 ([6, Example 4])**.** Give an example of a function satisfying the Hölder condition[2] with the power $\frac{1}{2}$, which is not a function of bounded variation in the sense Jordan. Do there exist functions satisfying the Hölder condition with the power $\alpha \in [0,1]$, which are not of bounded variation in the sense of Jordan, but they are not constant functions?

[2]Let us recall that a function $f\colon [a,b] \to \mathbb{R}$ satisfies the Hölder condition with the power $\alpha \in [0,1]$, if there exists such a constant $L > 0$ that $|f(x) - f(y)| \leq L|x-y|^\alpha$ for $x, y \in [a,b]$.

Chapter **5**

Almost periodic functions

Exercise 75 ([3, Zadanie 85, p. 42])**.** Prove that if there exists such a number $a \neq 0$ that

$$f(x+a) = \frac{1+f(x)}{1-f(x)} \qquad \text{for } x \in \mathbb{R}, \tag{75.1}$$

where $f\colon \mathbb{R} \to \mathbb{R} \setminus \{1\}$, then f is a periodic function.

Exercise 76 (cf. [3, Zadanie 35, p. 111] or [4, Exercise 5.16])**.** Prove that the unique nontrivial [1] solution to the following functional equation

$$f(x+y) + f(y-x) = 2f(x)f(y), \qquad x, y \in \mathbb{R}, \tag{76.1}$$

in the class of bounded and twice continuously differential functions on $\mathbb{R}$ is $f(x) = \cos ax$, where a is a given nonzero real number.

Exercise 77. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a periodic function. Prove that if f is differentiable, then the derivative f' is a periodic function. Is a primitive function of a continuous function a periodic one?

Exercise 78 ([3, Zadanie 259, p. 124] or [12, p. 119])**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function (locally integrable) and periodic of the period $\omega > 0$. Show that for an arbitrary $a \in \mathbb{R}$ and $n \in \mathbb{N}$ the following formula holds

$$\int_a^{a+n\omega} f(s)\mathrm{d}s = n \int_0^\omega f(s)\mathrm{d}s.$$

Exercise 79 ([18, Remark, p. 88])**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous periodic function of a period $\omega > 0$. Prove that

$$M(f) = \frac{1}{\omega} \int_0^\omega f(s)\mathrm{d}s.$$

Exercise 80. Show that a relatively dense set can be defined in an equivalent way using closed intervals as well as open intervals.

Exercise 81 ([17, p. 23] or [18, Example, p. 20])**.** Examine, which of the following subsets of the set of real numbers is relatively dense:

[1] Let us recall that a solution is said to be nontrivial if it is different from a constant function.

(a) $A = \{\pm k^2 : k \in \mathbb{N}_0\}$; (b) $B = \{\pm k^{\frac{1}{2}} : k \in \mathbb{N}_0\}$.

Exercise 82 (cf. [17, Przykład 1.1])**.** Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the formula $f(x) = \cos \alpha x + \cos \beta x$, where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ are incommensurate, is almost periodic, but it is not periodic.

Exercise 83 (cf. [18, Proposition 3, p. 26])**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be an almost periodic function. Prove that for arbitrary $a \in \mathbb{R}$ it holds $\|f\|_\infty = \sup_{x \geq a} |f(x)|$. Deduce that if an almost periodic function converges to zero as $x \to +\infty$, that is $\lim_{x \to +\infty} f(x) = 0$, then $f \equiv 0$.

Chapter 6

Appendix

Exercise 84 ([10, p. 14, 1.4.1, and p. 15, 1.4.2])**.** (a) Prove that any nondecreasing map $f\colon [0,1] \to [0,1]$ has a maximal fixed point.

(b) Show that if we additionally assume the leftside continuity, then there exists a minimal fixed point z_0 and $z_0 = \lim_{n\to\infty} f^n(0)$.

Exercise 85 ([10, p. 19, 1.6.26])**.** Let $\langle P, \preceq \rangle$ be a partially ordered set and $\mathcal{F}$ a nonempty commutative family of isotone mappings of P into itself. Assume that there exists a $b \in P$ such that $b \preceq f(b)$ for each $f \in \mathcal{F}$ and that every chain in $\{x \in P : b \preceq x\}$ has a supremum. Show that $\mathcal{F}$ has a maximal common fixed point

Exercise 86 ([10, p. 19, 1.6.19])**.** Let X, Y be two nonempty sets and $f\colon X \to Y$, $g\colon Y \to X$ two maps. Show that X and Y can be written as disjoint unions, $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, where $f(X_1) = Y_1$ and $g(Y_2) = X_2$. Derive the Cantor–Bernstein theorem.

Exercise 87 ([10, p. 169 and p. 34, 2.7.7])**.** Let A be a closed subset of a metric space X. Use the Knaster–Tarski theorem to show that A includes a maximal perfect subset.

Exercise 88 ([10, p. 34, 2.7.7])**.** Let C be a convex subset of a Hilbert space H and let $f\colon C \to C$ be nonexpansive. Show that the fixed point set of f is convex (possibly empty).

Exercise 89 ([10, p. 24, 2.1.6])**.** Let $f\colon \overline{B} \to H$ be a nonexpansive mapping of a closed ball in a Hilbert space H into that space. Show that if $(x|f(x)) \le \|x\|^2$ for every $x \in \partial\overline{B}$, then f has a fixed point.

Exercise 90. Give an example of an incomplete inner product space X and a nonexpansive mapping $f\colon \overline{B} \to \overline{B}$, where $\overline{B}$ is the closed unit ball in X, such that f has no fixed points.

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